BANACH SPACES WHICH CAN BE GIVEN AN EQUIVALENT UNIFORMLY CONVEX NORM

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ABSTRACT

Necessary and sufficient conditions for a Banach space to have an equivalent uniformly convex norm are given.

In this paper we give necessary and sufficient conditions for a Banach space to be isomorphic to a uniformly convex Banach space. Our main theorem, in fact, has quite a few corollaries which are of interest in Banach space theory since it shows the isomorphic equivalence between uniform convexity and uniform smoothness. Some examples of this are given at the end of this paper.

DEFINITIONS. An ordered pair (x_1, x_2) in a Banach space is a $(1,\varepsilon)$-part of a tree if $\| x_1 - x_2 \| \geqq \varepsilon$. Now if the (n,ε)-part of a tree is defined, we say that a 2^{n+1}-tuple $(x_1, x_2, \cdots, x_{2^{n+1}})$ is a $(n+1,\varepsilon)$-part of a tree if $\| x_{2j-1} - x_{2j} \| \geqq \varepsilon$, $1 \leqq j \leqq 2^n$ and the 2^n-tuple $(x_1 + x_2)/2$, $(x_3 + x_4)/2, \cdots, x_{2^{n+1}-1}/2 + x_{2^{n+1}}/2$ is an (n,ε)-part of a tree. Further, the Banach space B has the finite tree property, if there is an $\varepsilon > 0$ such that for every n, there is an (n,ε)-part of a tree where all elements have norm at most 1.

In James [1], it is proved that

A uniformly non-square Banach space does not have the finite tree property. (1)
This theorem combined with our main theorem below gives the corollaries listed at the end of this paper.

MAIN THEOREM. *A Banach space can be given an equivalent uniformly convex norm if and only if it does not have the finite tree property.*

PROOF. The "only-if" part of this theorem is easy (a proof can be found in [1]) and so we will only prove the "if" part. This will follow from the lemmas below. In order to state our first lemma we have to make some

DEFINITIONS. Let B be a Banach space and $z \in B$. The ordered pair (x_1, x_2) is a $(1,\varepsilon)$-partition of z if $x_1 + x_2 = z$, $\|x_1\| = \|x_2\|$ and $\left\| \frac{x_1}{\|x_1\|} - \frac{x_2}{\|x_2\|} \right\| \geqq \varepsilon$. Having defined an (n,ε)-partition of z, we say that the 2^{n+1}-tuple $(y_1, y_2, \cdots, y_{2^{n+1}})$ is an $(n+1,\varepsilon)$-partition of z if $\|y_{2j-1}\| = \|y_{2j}\|$, $\left\| \frac{y_{2j-1}}{\|y_{2j-1}\|} - \frac{y_{2j}}{\|y_{2j}\|} \right\| \geqq \varepsilon$, $1 \leqq j \leqq 2^n$, and the 2^n-tuple $(y_1 + y_2, y_3 + y_4, \cdots, y_{2^{n+1}-1} + y_{2^{n+1}})$ is an (n,ε)-partition of z. If $(x_1, x_2, \cdots, x_{2^n})$ is an (n,ε)-partition of z, then the (k,ε)-partition $(x_1 + x_2 + \cdots + x_{2^{n-k}}, x_{2^{n-k}+1} + x_{2^{n-k}+2} + \cdots + x_{2^{n-k+1}}, \cdots, x_{2^n - 2^{n-k}+1} + x_{2^n-2^{n-k}+2} + \cdots + x_{2^n})$ is the k-part of the (n,ε)-partition $(x_1, x_2, \cdots, x_{2^n})$. For simplicity we say that (z) is a $(0,\varepsilon)$-partition of z.

LEMMA 1. *If a Banach space B does not have finite tree property, then for every $\varepsilon > 0$ there is an n and a δ, $\delta > 0$, such that if $z \in B$ and $(x_1, x_2, \cdots, x_{2^n})$ is an (n,ε)-partition of z, then*

$$\sum_{j=1}^{2^n} \|x_j\| \geqq (1+\delta)\|z\|.$$

PROOF. Assume $\|z\| = 1$ and let $(x_1, x_2, \cdots, x_{2^m})$ be an (m,ε)-partition of z. If we take the 1-part $(x_1 + x_2 + \cdots + x_{2^{m-1}}, x_{2^{m-1}+1} + x_{2^{m-1}+2} + \cdots + x_{2^m})$ of this partition and multiplity the 2 vectors of it by 2, we get (by the definition of (m,ε)-partition and $(1,\varepsilon)$-part of a tree) a $(1,\varepsilon)$-part of a tree where both of the vectors have length $\geqq 1$. Inductively we see, that if we take the k-part of the (m,ε)-partition and multiply the vectors of it by 2^k, then we will get a (k,ε)-part of a tree where all of the vectors have norm $\geqq 1$. Now since B does not have the finite tree property there is an n and a δ, $\delta > 0$, such that for all such (n,ε)-parts of trees formed by multiplication by 2^n of the vectors in an (n,ε)-partition, there is one vector of length $\geqq 1 + 2^n\delta$. And this gives directly

$$\sum_{j=1}^{2^n} \|x_j\| \geqq 1 + \delta = (1+\delta)\|z\|.$$

For the next lemma we need

DEFINITION. We say that a real-valued function $x \to |x|$ is an ecart in B if

a) $|x| \geqq 0$ for all x and $|x| = 0 \Leftrightarrow x = 0$

b) $|\alpha x| = |\alpha| \; |x|$ for all x and real α.

LEMMA 2. *Let B be a Banach space which does not have the finite tree property. Let ε be a positive number and let n and δ be as in the conclusion of Lemma 1. Assume $0 < \delta < \varepsilon < {}^1/_8$. Then there is an ecart in B and a $\delta_1 > 0$ such that*

a) $(1-\delta)\|x\| \leqq |x| \leqq \left(1 - \frac{\delta}{3}\right)\|x\|$

b) *if* $\|x\| = \|y\| = 1$ *and* $\|x-y\| \geqq \varepsilon$ *then* $|x+y| < |x| + |y| - \delta_1$.

PROOF. For each z and each (m,ε)-partition $(u_1, u_2, \cdots, u_{2^m})$, $0 \leqq m \leqq n$, of z we consider the number

$$\frac{\sum_{j=1}^{2^m} \|u_j\|}{1 + \frac{\delta}{2}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^m}\right)}.$$

For a fixed z, we denote the infimum of these numbers by $|z|$. Obviously this is an ecart. We also see directly that $(1-\delta)\|z\| \leqq |z| \leqq (1-\delta/3)\|z\|$; the right inequality is obtained from the $(0,\varepsilon)$-partition of z. It follows from Lemma 1 that the infimum does not change if we consider (m,ε)-partitions with $0 \leqq m \leqq n-1$. Now let $\|x\| = \|y\| = 1$ and $\|x-y\| \geqq \varepsilon$ hold.

Then (x,y) is a $(1,\varepsilon)$-partition of $x+y$. Choose γ, $0 < \gamma < \delta/4^{2n}$, and a (k,ε)-partition $(u_1, u_2, \cdots, u_{2^k})$ of x and an (l,ε)-partition $(w_{l,1}, w_{l,2}, \cdots, w_{l,2^l})$ of y such that

$$|x| > \frac{\sum_{j=1}^{2^k} \|u_j\|}{1 + \frac{\delta}{2}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^k}\right)} - \gamma \text{ and } |y| > \frac{\sum_{1=j}^{2^l} \|w_{l,j}\|}{1 + \frac{\delta}{2}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^l}\right)} - \gamma.$$

We can assume $n - 1 \geqq l \geqq k \geqq 0$. We denote by $(w_{k,1}, w_{k,2}, \cdots, w_{k,2^k})$ the k-part of the (l,ε)-partition of y. We then have

$$|y| > \frac{\sum_{j=1}^{2^l} \| w_{l,j} \|}{1 + \frac{\delta}{2}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^l}\right)} - \gamma \geqq \frac{\sum_{j=1}^{2^k} \| w_{k,j} \|}{1 + \frac{\delta}{2}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^l}\right)} - \gamma$$

$$\geqq \frac{\sum_{j=1}^{2^k} \| w_{k,j} \|}{1 + \frac{\delta}{2}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^{k+1}} + \frac{1}{3 \cdot 4^{k+1}}\right)} - \gamma.$$

We now consider the $(k + 1,\varepsilon)$-partition $(u_1, u_2, \cdots, u_{2^k}, w_{k,1}, w_{k,2}, \cdots, w_{k\,2^k})$ of $x + y$ which we obtain from the (k,ε)-partition of x and the k-part of the (l,ε)-partition of y.

This gives

$$|x + y| \leqq \frac{\sum_{j=1}^{2^k} \| u_j \| + \sum_{j=1}^{2^k} \| w_{k,j} \|}{1 + \frac{\delta}{2}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^{k+1}}\right)}.$$

The inequalities which we have obtained for x, y and $x + y$ now give

$$|x| + |y| - |x + y| \geqq \sum_{j=1}^{2^k} \|u_j\| \left(\frac{1}{1 + \frac{\delta}{2}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^k}\right)} - \frac{1}{1 + \frac{\delta}{2}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^{k+1}}\right)} \right)$$

$$- \sum_{j=1}^{2^k} \|w_{k,j}\| \left(\frac{1}{1 + \frac{\delta}{2}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^{k+1}}\right)} - \frac{1}{1 + \frac{\delta}{2}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^{k+1}} + \frac{1}{3 \cdot 4^{k+1}}\right)} \right) - 2\gamma$$

Since $\sum_{j=1}^{2^k} \| u_j \|$ and $\sum_{j=1}^{2^k} \| w_{k,j} \|$ are both between 1 and $1 + \delta$, this is

$$\geqq 1 \cdot \frac{\frac{\delta}{2} \cdot \frac{1}{4^{k+1}}}{\left(1 + \frac{\delta}{2}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^k}\right)\right)\left(1 + \frac{\delta}{2}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^{k+1}}\right)\right)}$$

$$-(1 + \delta) \cdot \frac{\frac{\delta}{2} \cdot \frac{1}{3 \cdot 4^{k+1}}}{\left(1 + \frac{\delta}{2}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^{k+1}}\right)\right)\left(1 + \frac{\delta}{2}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^{k+1}} + \frac{1}{3 \cdot 4^{k+1}}\right)\right)} - 2\gamma$$

and since $\gamma < \frac{\delta}{4^{2n}}$ this is $\geqq \frac{\delta}{2} \cdot \frac{1}{4^{k+3}} \geqq \frac{\delta}{2} \cdot \frac{1}{4^{n+2}}$ since $0 < \delta < \frac{1}{8}$. The lemma is proved.

LEMMA 3. *Let B be a Banach space with norm $\| \ \|$, which does not have the finite tree property. Let ε be a positive number and let n and δ be as in the conclusion of Lemma 1. Assume $0 < \delta < \varepsilon < 1/8$. Then it is possible to introduce a norm $\| \ \|_{5\varepsilon}$ in B such that*

a) $(1 - \delta) \| x \| \leqq \| x \|_{5\varepsilon} \leqq (1 - \delta/3) \| x \|$.

b) $\| x \| = \| y \| = 1$ *and* $\| x - y \| \geqq 5\varepsilon \Rightarrow \| x + y \|_{5\varepsilon} \leqq \| x \|_{5\varepsilon} + \| y \|_{5\varepsilon} - \varepsilon\delta_1$, *where δ_1 is the same as in the conclusion of Lemma 2.*

PROOF. We introduce an ecart $| \ |$ in B as in Lemma 2, and then we let $\| x \|_{5\varepsilon}$ be the infimum of the lengths in this ecart of polygons connecting 0 and x. Then $\| \ \|_{5\varepsilon}$ is obviously a norm which satisfies *a*). Now assume that $\| x \| = \| y \| = 1$ and $\| x - y \| \geqq 5\varepsilon$. Choose $\gamma > 0$ so that $\delta + \gamma < \varepsilon$. Let $0 = x_0, x_1, x_2, \cdots, x_n = x$ and $0 = y_0, y_1, y_2, \cdots, y_m = y$ be two polygons of lengths $\leqq \| x \|_{5\varepsilon} + \gamma$ and $\| y \|_{5\varepsilon} + \gamma$ in the ecart. Then they have lengths between 1 and $1 + \delta + \gamma < 1 + \varepsilon$ in $\| \ \|$. Let us assume that in $\| \ \|$ the length of $0, x_1, x_2, \cdots, x_n$ is not larger than that of $0, y_1, y_2, \cdots, y_m$. By introducing new division points, at most m in the polygon for x and at most n in the polygon for y, it is easy to have in the new polygons $\| x'_1 \| = \| y'_1 \|$, $\| x'_k - x'_{k-1} \| = \| y'_k - y'_{k-1} \|$ for all those k where x'_k is defined. If $x'_{k_1} = x$, then the length in $\| \ \|$ of the polygon connecting y'_{k_1} and y will be $< \varepsilon$. Thus $\| x - y'_{k_1} \| \geqq 4\varepsilon$, and we have the inequality

$$\begin{aligned} 4\varepsilon \leqq \| x - y'_{k_1} \| &= \Big\| \sum_{k=1}^{k_1} [(x'_k - x'_{k-1}) - (y'_k - y'_{k-1})] \Big\| \\ &\leqq \sum_{k=1}^{k_1} \| (x'_k - x'_{k-1}) - (y'_k - y'_{k-1}) \| . \end{aligned}$$

This gives that $\Sigma_j \| x'_{k_j} - x'_{k_j - 1} \| \geqq \varepsilon$ where the summation is extended over all those k_j where $\| (x'_{k_j} - x'_{k_j-1}) - (y'_{k_j} - y'_{k_j-1}) \| \geqq \varepsilon \| x'_{k_j} - x'_{k_j-1} \|$. For all these k_j's we have by Lemma 2

$$\begin{aligned} | (x'_{kj} - x'_{k_j-1}) + (y'_{k_j} - y'_{k_j-1}) | \leqq | x'_{k_j} - x'_{k_j-1} | + | y'_{k} - y'_{k_j-1} | \\ - \| x'_{k_j} - x'_{k_j-1} \| \cdot \delta_1 . \end{aligned}$$

This gives that there is a polygon connecting 0 and $x+y$ which has length $\leqq \|x\|_{5\varepsilon}+\|y\|_{5\varepsilon}+2\gamma-\varepsilon\cdot\delta_1$ in the ecart. Since γ is arbitrarily small the lemma is proved.

LEMMA 4. *Let B be a linear space with two norms $\|\ \|$ and $\|\ \|_u$. Assume that for all $x\in B$, $\|x\|_u\leqq\|x\|\leqq 2\|x\|_u$. Assume also that there exists a real-valued function $\delta(\varepsilon)$, $\delta(\varepsilon)>0$ if $\varepsilon>0$, such that if $\|x\|=\|y\|=1$ and $\|x-y\|\geqq\varepsilon$, then $\|x+y\|_u\leqq\|x\|_u+\|y\|_u-\delta(\varepsilon)$. Then $\|\ \|_u$ is a uniformly convex norm.*

PROOF. We first observe that, for any norm $\|\ \|_1$ on a linear space, we have that $\|x_1\|=\|y\|_1=1$ and $1\geqq\|x-y\|_1\geqq\varepsilon>0$ implies $\inf\|\alpha x-y\|_1\geqq\varepsilon/_2$.$(a)$

We prove (a) in the following way: For $|\alpha-1|\leqq\varepsilon/2$, (a) follows from the inequality $\|\alpha x-y\|_1\geqq\|x-y\|_1-|\alpha-1|\cdot\|x\|_1\geqq\varepsilon-\varepsilon/2=\varepsilon/2$.

For $0\leqq\alpha\leqq 1-\varepsilon/2$ or $1+\varepsilon/2\leqq\alpha$, (a) follows from the inequality $\|\alpha x-y\|_1\geqq|\,\|\alpha x\|_1-\|y\|_1|=|\alpha-1|\geqq\varepsilon/2$. Finally, for $\alpha\leqq 0$, (a) follows from the fact that $\|\alpha x-y\|$ is a convex function of α which by assumption takes the value 1 at $\alpha=0$ and some value $\leqq 1$ at $\alpha=1$.

Now assume that $\|x\|_u=\|y\|_u=1$ and $1/10\geqq\|x-y\|_u\geqq\varepsilon$. Then for some pair $\alpha,\beta,\frac{1}{2}\leqq\alpha\leqq 1$, $\frac{1}{2}\leqq\beta\leqq 1$, we have $\|\alpha x\|=\|\beta y\|=1$. By (a), we have $\|\alpha x-\beta y\|\geqq\|\alpha x-\beta y\|_u\geqq\frac{1}{2}\cdot\varepsilon/2$. Thus by the assumptions of the lemma, we have $\|\alpha x+\beta y\|_u\leqq|\alpha|+|\beta|-\delta(\varepsilon/4)$. This gives $\|x+y\|_u\leqq\|\alpha x+\beta y\|_u+\|(1-\alpha)x+(1-\beta)y\|_u\leqq|\alpha|+|\beta|-\delta(\varepsilon/4)+(1-\alpha)+(1-\beta)=2-\delta(\varepsilon/4)$ and so we have proved that $\|\ \|_u$ is a uniformly convex norm.

LEMMA 5. *If in a Banach space B for every $\varepsilon>0$ it is possible to introduce a norm satisfying a) and b) of Lemma 3, then B can be given a uniformly convex norm.*

PROOF. Put, with the notations of Lemma 3

$$\|x\|_u=\frac{1}{2}\|x\|_\varepsilon+\frac{1}{4}\|x\|_{\varepsilon/2}+\frac{1}{8}\|x\|_{\varepsilon/4}+\cdots.$$

Then $\|x\|_u$ is obviously a norm on B and $(1-\varepsilon)\|x\|\leqq\|x\|_u\leqq\|x\|$. Now assume that $\|x\|=\|y\|=1$ and that $\|x-y\|\geqq\varepsilon_1$. Then $\varepsilon_1>\varepsilon/2^k$ for some k and so it follows from Lemma 3 and the definition of $\|x\|_u$ that $\|x+y\|_u\leqq\|x\|_u$

$+ \| y \|_u - 1/2^{k+1} \cdot \varepsilon/2^k \cdot \delta_1, \varepsilon/5 \cdot 2^k$ where $\delta_1, \varepsilon/5 \cdot 2^k$ is the δ_1 of Lemma 3 corresponding to $\varepsilon/2^k$. This gives by Lemma 4, that $\| \quad \|_u$ is a uniformly convex norm on B. Thus the lemma and the main theorem are proved.

We now list some corollaries of our main theorem.

COROLLARY 1. *A Banach space can be given an equivalent uniformly convex norm if and only if it can be given an equivalent uniformly smooth norm.*

PROOF. If a Banach space can be given a uniformly smooth norm then this norm will be uniformly non-square. Thus by (1) and our main theorem it can be given a uniformly convex norm. The "only-if" part follows by duality.

It has been proved by E. Asplund [5] that if a Banach space can be given an equivalent uniformly convex norm and an equivalent uniformly smooth norm, then it can be given a norm which is both uniformly smooth and uniformly convex. We thus obtain

COROLLARY 2. *If a Banach space can be given an equivalent uniformly non-square norm, then it can be given an equivalent norm, which is both uniformly convex and uniformly smooth.*

The concept "super-reflexive space" has been introduced by James, who has proved theorems for these spaces (see [2] and [3]). It is a theorem of James that a Banach space is super-reflexive if it is isomorphic to a space which is either uniformly convex or uniformly non-square. So our main theorem and (1) now give

COROLLARY 3. *A Banach space can be given a uniformly convex norm if and only if it is super-reflexive.*

It is a known result (see Day [4], pp. 113–114) that uniform convexity of a Banach space is dual to uniform Frechet differentiability of the norm so we also get

COROLLARY 4. *A Banach space can be given a uniformly Frechet differentiable norm if and only if its dual can be given such a norm.*

REMARK. The following result follows directly from our main theorem and the results of (1): If it is true that a Banach space, which is uniformly non-$l_1(n)$ for some n, is reflexive, then it is true that such a Banach space can be given an equivalent uniformly convex norm.

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